Amalgamated Worksheet # 4

Various Artists

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Complex Spectral Theorem Let V be a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Then V has an orthonormal basis of eigenvectors of T if and only if T is normal

Real Spectral Theorem Let V be a vector space over $\mathbb{R}$ and $T \in \mathcal{L}(V)$. Then V has an orthonormal basis of eigenvectors of T if and only if T is self-adjoint

Problem 1:

Suppose V is a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Show that if every eigenvalue of T is real, then T is self-adjoint.

Problem 2:

Suppose A is a symmetric matrix over $\mathbb{R}$. Show that if $|\lambda| = 1$ for every eigenvalue λ of A, then $A^2 = I$.

Problem 3:

(if time permits) If $U = \mathcal{P}_2(\mathbb{R})$, what is its complexification $U_{\mathbb{C}}$? Suppose $S \in \mathcal{L}(U)$ is defined by $S(p) = p'$, what is its complexification $S_{\mathbb{C}}$?

Problem 4:

Briefly explain how to prove the real spectral theorem from the complex spectral theorem

2 Daniel Sparks

1. Prove some of the properties of adjoints listed on p.119; S, T are operators on a finite dimensional complex vector space V.

   (a) $(S+T)^* = S^* + T^*$

   (b) $(aT)^* = \overline{a}T^*$

   (c) $(T^*)^* = T$

   Axler suggests thinking about $T \mapsto T^*$ as a function $* : \mathcal{L}(V) \to \mathcal{L}(V)$.

   (f) Look up the definition of a $\mathbf{C}^*$-algebra.

   (g) Show that $*$ is an isomorphism of $\mathbf{R}$ vector spaces.

2. Prove in detail DWD Lemma 7.1. Namely, if T is normal on a finite dimensional complex vector space V, then $\text{Null}(T) = \text{Null}(T^*)$.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 complex matrix. This determines a linear map $L_A : \mathbf{C}^2 \to \mathbf{C}^2$ by $v \mapsto Av$. Using only the identity

$$\langle L_A v, w \rangle = \langle v, L_A^* w \rangle$$

   show that $L_A^* = L_{\overline{A}^t}$, where $\overline{A}^t = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$. "The adjoint of a matrix operator is the conjugate transpose." (For the ambitious, try an $n \times n$.)

4. Fill in the blanks. Let T be a normal operator on a finite dimensional $\mathbf{C}$-vector space V. Consider the list of distinct eigenvalues of T: $\lambda_1, \cdots, \lambda_m$. (This list is nonempty becauase (a) __________.)

   Let U_{λ_i} be the (b) __________ corresponding to λ_i. By (c) __________, we have a decomposition $V = U_{\lambda_1} \oplus \cdots \oplus U_{\lambda_m}$.

   Let $e_i = \dim U_{\lambda_i}$ be the (d) __________ of λ_i. Then we have bases $\beta_i' = \{u_{i,1}', \cdots, u_{i,e_i}'\}$ for each U_{λ_i}. We may use the (e) __________ to obtain orthonormal bases $\beta_i = \{u_{i,1}, \cdots, u_{i,e_i}\}$ of each U_{λ_i}. We know that the concatenated list $\beta = (\beta_1, \cdots, \beta_m) = \{u_{1,1}, u_{1,2} \cdots, u_{1,e_1}, u_{2,1}, \cdots, u_{m,e_m}\}$ is a basis for V because (f) __________. Notice that each of these basis vectors are normal (i.e. of norm 1) generalized eigenvectors, and that $u_{i,j} \perp u_{k,l}$ whenever $i = k$.

   Now, since T is normal, because (g*) __________ we know that β is actually a basis of eigenvectors. Finally, because (h*) __________ we know that $u_{i,j} \perp u_{k,l}$ whenever $i \neq k$. Therefore β is an orthonormal eigenbasis.

   * These results do not have names, but can be found in DWD.

5. Review/redo carefully the proofs of the results cited in (g) and (h) of the previous exercise.